Estimation of dynamic discrete choice models

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We start with an single-agent models of dynamic decisions:

- Machine replacement and investment decisions: Rust (1987)
- Renewal or exit decisions: Pakes (1986)

This lecture will focus on econometrics methods, and next lecture will discuss mostly applications.

Next, we will discuss questions related to the dynamic of industries:

- Markov-perfect dynamic games
- Empirical model of static and dynamic games

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1 These lectures notes incorporate material from Victor Agguirregabiria’s graduate IO slides at the University of Toronto.
Machine replacement and investment decisions

- Consider a firm producing a good at $N$ plants (indexed by $i$) that operate independently.
- Each plant has a machine.
- Examples:
  - Rust (1987): Each plant is a Madison WI bus, and Harold Zucher is the plant operator.
  - Das (1992): Consider cement plants, where the machines are cement kiln.
  - Rust and Rothwell (1995): Study the maintenance of nuclear power plants.
- Related applications: Export decisions (Das et al. (2007)), replacement of durable goods (Adda and Cooper (2000), Gowrisankaran and Rysman (2012)).
Bus Replacement: Rust (1987)

- Profit function at time $t$:
  \[ \pi_t = \sum_{i=1}^{N} y_{it} - r_{Ci_t} \]

  where $y_{it}$ is the plant’s variable profit, and $r_{Ci_t}$ is the replacing cost of the machine.

- Replacement and depreciation:
  - Replace cost:
    \[ r_{Ci_t} = a_{it} \times RC(x_{it}) \]
    where $\partial RC(x)/\partial x \geq 0$ and $a_{it} = 1$ if the machine is replaced. In the application, $RC(x_{it}) = \theta_{R_0} + \theta_{R_1} x_{it}$.
  - State variable: machine age $x_{it}$, choice-specific profit shock $\{\epsilon_{it}(0), \epsilon_{it}(1)\}$.
  - Variable profits are decreasing in the age $x_{it}$ of the aging, and increasing in profit shock $\epsilon_{it}(a_{it})$:
    \[ y_{ij} = Y((1 - a_{it})x_{it}, \epsilon_{it}(a_{it})) \]
    where $\partial Y/\partial x < 0$. 

Estimation of dynamic discrete choice models
Profits and Depreciation

- **Variable profit: Step function**
  \[
  \pi_{it} = \begin{cases} 
  Y(0, \epsilon_{it}(1)) - RC(x_{it}) & \text{If } a_{it} = 1 \\
  Y(x_{it}, \epsilon_{it}(0)) & \text{Otherwise.}
  \end{cases}
  \]

- **Aging/depreciation process:**
  - **Deterministic:** \( x_{it+1} = (1 - a_{it})x_{it} + 1 \)
  - **Stochastic:** \( x_{it+1} = (1 - a_{it})x_{it} + \xi_{t+1} \)

**Note:** In Rust (1987), \( x_{it} \) is bus mileage. It follows a random walk process with a log-normal distribution.

**Assumptions:**

1. **Additive separable (AS) profit shock:**
   \[ Y((1 - a)x, \epsilon(a)) = \theta Y_0 + \theta Y_1 (1 - a)x + \epsilon(a) \]

2. **Conditional independence (CI):** \( f(\epsilon_{t+1}|\epsilon_t, x_t) = f(\epsilon_{t+1}) \)

3. **Aging follows is a discrete random-walk process:** \( x_{it} \in \{0, 1, ..., M\} \) and matrix \( F(x'|x, a) \) characterizes its controlled Markov transition process.
Dynamic Optimization

- Harold Zucher maximizes expected future profits:

\[ V(a_{it}|x_{it}, \epsilon_{it}) = E \left( \sum_{\tau=0}^{\infty} \beta^\tau \pi_{it+\tau} \bigg| x_{it}, \epsilon_{it}, a_{it} \right) \]

- **Recursive formulation**: Bellman equation

\[
V(a|x, \epsilon) = Y((1 - a) \cdot x) - RC(a \cdot x) + \epsilon(a) \\
+ \beta \sum_{x'} E_{\epsilon'} \left( V(x', \epsilon') \right) F(x'|x, a) \\
= v(a, x) + \epsilon(a)
\]

where \( V(x, \epsilon) \equiv \max_{a \in \{0,1\}} V(a|x, \epsilon) \).

- Optimal replacement decision:

\[ a^* = \begin{cases} 
1 & \text{If } v(1, x) - v(0, x) = \bar{v}(x) > \epsilon(0) - \epsilon(1) = \bar{\epsilon} \\
0 & \text{Otherwise.}
\end{cases} \]

- If \( \{\epsilon(0), \epsilon(1)\} \) are distributed according to a T1EV distribution with unit variance:
Solution to the dynamic-programming (DP) problem

- Assumptions (1) and (2) imply that we only need numerically find a fixed-point to the “Emax” function $\bar{V}(x)$ ($M$ elements):

$$
\bar{V}(x) = E_\epsilon \left( \max_a v(a, x) + \epsilon(a) \right) \\
= E_\epsilon \left( \max_a \Pi(a, x) + \beta \sum_{x'} \bar{V}(x') F(x'|x, a) + \epsilon(a) \right) \\
= \Gamma(x|\bar{V})
$$

where $\Pi(a, x) = Y((1 - a) \cdot x) - RC(a \cdot x)$, and $\Gamma(x|\bar{V})$ is a contraction mapping.

- Matrix form representation using the T1EV distribution assumption:

$$
\bar{V} = \ln \left( \exp \left( \Pi(0) + \beta F(0)\bar{V} \right) + \exp \left( \Pi(1) + \beta F(1)\bar{V} \right) \right) + \gamma \\
= \Gamma(\bar{V})
$$

where $\gamma$ is the Euler constant, $F(0)$ and $F(1)$ are two $M \times M$ conditional transition probability matrix.
Algorithm 1: Value Function Iteration

- **Fixed objects:**
  - Payoffs ($M \times 1$):
    \[
    \Pi(a) = \{\theta_0 + \theta_x(1-a)x_i - RC(a \cdot x)\}_{i=1,...,M} \text{ for } a \in \{0, 1\}
    \]
  - Conditional transition probability ($M \times M$): $F(a)$ for $a \in \{0, 1\}$
    \[
    F_{j,k}(a) = F(x_{t+1} = x_k| x_t = x_j, a_t = a)
    \]
  - Stopping rule: $\eta \approx 10^{-14}$.

- **Value function iteration algorithm:**
  1. Guess initial value for $\bar{V}^0(x)$. Example: Static value function
    \[
    \bar{V}^0(x) = \ln (\exp (\Pi(0)) + \exp (\Pi(1))) + \gamma
    \]
  2. Update value function iteration $k$:
    \[
    \bar{V}^k = \ln \left( \exp (\Pi(0) + \beta F(0)\bar{V}^{k-1}) + \exp (\Pi(1) + \beta F(1)\bar{V}^{k-1}) \right) + \gamma
    \]
  3. Stop if $||\bar{V}^k - \bar{V}^{k-1}|| < \eta$. Otherwise, repeat steps (2)-(3).
Policy Function Representation

- Define conditional choice-probability (CCP) mapping:

\[
P(x) = \Pr \left( \frac{\Pi(1, x) + \beta \sum_{x'} \tilde{V}(x') F(x'|x, 1) + \epsilon(1)}{\geq \Pi(0, x) + \beta \sum_{x'} \tilde{V}(x') F(x'|x, 0) + \epsilon(0)} \right) \tag{1}
\]

\[
= \exp(\tilde{v}(x)/(1 + \exp(\tilde{v}(x)))) = (1 + \exp(-\tilde{v}(x)))^{-1}
\]

Where, \( \tilde{v}(x) = v(1, x) - v(0, x) \).

- At the “optimal” CCP, we can write the Emax function as follows:

\[
\tilde{V}^P(x) = (1 - P(x)) \left[ \Pi(0, x) + e(0, x) + \beta \sum_{x'} \tilde{V}^P(x') F(x'|x, 0) \right]
\]

\[
+ P(x) \left[ \Pi(0, x) + e(1, x) + \beta \sum_{x'} \tilde{V}^P(x') F(x'|x, 1) \right]
\]

where \( e(a, x) = E(\epsilon(a)|a^* = a, x) \) is the conditional expectation \( \epsilon(a) \).
Policy Function Representation (continued)

- If $\epsilon(a)$ is T1EV distributed, we can write this expectation analytically:

$$e(a, x) = \gamma - \ln P(a|x).$$

- This implicitly define the value function in terms of the CCP vector:

$$\bar{V}^P = (I - \beta F^P)^{-1} \left[ (1 - P) \ast (\Pi(0) + e(0)) + P \ast (\Pi(1) + e(1)) \right]$$

where $F^P = (1 - P) \ast F(0) + P \ast F(1)$ and $\ast$ is the element-by-element multiplication operator.

- Equations 1 and 2 define a fixed-point in $P$:

$$P^* = \Psi(P^*)$$

where $\Psi(\cdot)$ is a contraction mapping.
Algorithm 2: Policy Function Iteration

1. Guess initial value for the CCP. Example: Static choice-probability

   \[ P(x) = (1 + \exp(- (\Pi(x|1) - \Pi(x|0))))^{-1} \]

2. Calculate expected value function:

   \[ \tilde{V}^{k-1} = \left( I - \beta F^{k-1} \right)^{-1} \left[ \begin{array}{c} (1 - P^{k-1}) \ast (\Pi(0) + e^{k-1}(0)) \\ + P^{k-1} \ast (\Pi(1) + e^{k-1}(1)) \end{array} \right] \]

3. Update CCP:

   \[ P^k(x) = \Psi(P^{k-1}(x)) = \left( 1 + \exp(- \tilde{v}^{k-1}(x)) \right)^{-1} \]

   where \( \tilde{v}^{k-1} = (\Pi(1) + \beta F(1)\tilde{V}^{k-1}) - (\Pi(0) + \beta F(0)\tilde{V}^{k-1}) \).

4. Stop if \( \|P^k - P^{k-1}\| < \eta \). Otherwise, repeat steps (2)-(4).
Value-function *versus* Policy-function Algorithms

- Both algorithms are guaranteed to converge if $\beta \in (0, 1)$
- Policy-function iteration algorithms converges in fewer steps than value-function iteration.
- However, each step of the policy-function algorithm is *slower* due to the matrix inversion. $M$ is typically very large (in the millions).
- If $M$ is very large, it can be faster and more accurate to find $\bar{V}$ using linear programing tools (e.g. linsolve in Matlab):

$$
(I - \beta F^{k-1}) \bar{V}^{k-1} = (1 - P^{k-1}) \ast (\Pi(0) + e^{k-1}(0)) + P^{k-1} \ast (\Pi(1) + e^{k-1}(1))
\Leftrightarrow Ay = b
$$

- Suggested algorithm:
  - Start with value-function iteration if $\bar{V}^k(x) - \bar{V}^{k-1}(x) > \eta^1$
  - Switch to policy-function iteration when $\bar{V}^k(x) - \bar{V}^{k-1}(x) < \eta^1$
  - Where $\eta^1 < \eta$ (e.g. $\eta^1 = 10^{-2}$)
Estimation: Nested fixed-point MLE

- **Data:** Panel of choices $a_{it}$ and observed states $x_{it}$
- **Parameters:** Technology parameters $\theta = \{\theta_{Y_0}, \theta_{Y_1}, \theta_{R_0}, \theta_{R_1}\}$, discount factor $\beta$, and distribution of mileage shocks $f_x(\xi_{it})$.
- **Initial step:** If the panel is long-enough, we can estimate $f_x(\xi)$ from the data. The estimated process can then be discretized to construct $\hat{F}(1)$ and $\hat{F}(0)$.
- **Maximum likelihood problem:**

  $$\max_{\theta, \beta} \sum_i \sum_t a_{it} \ln P(x_{it}) + (1 - a_{it}) \ln (1 - P(x_{it}))$$

  s.t. $P(x_{it}) = \Psi(x_{it}) \quad \forall x_{it}$

- In practice, we need two functions:
  - **Likelihood:** Evaluate $L(\theta, \beta)$ given $P(x_{it})$.
  - **Fixed-point:** Routine that solves $P(x_{it})$ for every guess of $\theta, \beta$. 
Incorporating Unobserved Heterogeneity

- **Why?** Relax the conditional independence assumption.
- **Example:** Buses have heterogeneous replacement costs ($K$ types)
  - This increases the number of parameters by $K(K-1)$: \{\theta^1_{R_0}, \ldots, \theta^K_{R_0}\} + \{\omega_1, \ldots, \omega_{K-1}\}$ (probability weights).
  - E.g.: discretize a parametric distribution: $\ln \theta^i_{R_0} \sim \mathcal{N}(\mu, \sigma^2)$
  - This changes the MLE problem:

\[
\max_{\theta, \beta, \omega} \sum_i \ln \left[ \sum_k g(k|x_{i1}) \prod_t P_k(x_{it})^{a_{it}} (1 - P_k(x_{it}))^{1-a_{it}} \right]
\]

s.t. \( P_k(x_{it}) = \Psi_k(x_{it}) \quad \forall x_{it} \) and type $k$

Where $g(k|x_{i1})$ is the probability that bus $i$ is type $k$ conditional on initial milage $x_{i1}$ (i.e. initial condition problem).
- How to calculate $g(k|x_{i1})$?
Side note: The initial condition problem

- Unobserved heterogeneity creates a correlation between the initial state (i.e. $x_{i1}$ mileage) and types (Heckman 1981).
- Two solutions:
  - **New buses**: Exogenous initial assignment $g(k|x_{i1}) = \omega_k$.
  - **Limiting distribution**: The bus engine replacement creates a finite-state Markov chain defined by
    \[
    F_k(x'|x) = \sum_a P_k(a|x)F(x'|x, a) \text{ for each type } k
    \]

Under fairly general assumptions, this process generates a unique limiting distribution:

\[
\pi_k(x) = \sum_{i=1}^{M} F_k(x_{t+1} = x|x_t = x_i)\pi_k(x_i) \leftrightarrow \pi_k = F_k^T \pi_k
\]

We can use the limiting distribution to calculate the type probability conditional on initial mileage:

\[
g(k|x_{i1}) = \frac{\omega_k \pi_k(x_{i1})}{\sum_{k'} \omega_{k'} \pi_{k'}(x_{i1})}
\]
Identification: Residual profit

- **Assumption:** Parametric distribution function $F_\epsilon$.
- Standard normalization: $\sigma_\epsilon = 1$.
  - This means that we cannot identify the “dollar” value of replacement costs. Only relative to variable profits.
  - True in any discrete-choice problem.
- When profits or output data are available, we can relax this normalization, and estimate $\sigma_\epsilon$ (e.g. investment and production data).
Identification: Discount Factor

- The data is summarized by the empirical hazard function:

\[ h(x) = \Pr(\text{replacement}_t | \text{miles}_t = x) \]

- This corresponds to the reduced form of the model:

\[
\begin{align*}
    h(x) &= P(x) = F_{\tilde{\epsilon}}(\tilde{\nu}(x)) \\
    &= F_{\tilde{\epsilon}} \left( -\beta \sum_{x'} V(x')(F(x'|x, 1) - F(x'|x, 0)) \right) \\
\end{align*}
\]

- **Claim:** \( \beta \) is not identified, unless we parametrize payoffs: \( Y \) and \( RC \).
  - If \( \Pi(x) \) is linear in \( x \), then non-linearity in the observed hazard function identifies \( \beta \).
  - If \( \Pi(x) \) is a non-parametric function, we cannot distinguish between a non-linear myopic model (\( \beta = 0 \)), and a forward-looking model (\( \beta > 0 \)).

- **What would identify \( \beta \)?**
  - **Exclusion restriction:** The model includes a state variable \( z \) that only enters the Markov transition function (i.e. \( F(x'|x, z, a) \)), and not the static payoff function.
Identification of $\beta$ and search for the right specification

### TABLE VIII
**Summary of Specification Search**

<table>
<thead>
<tr>
<th>Cost Function</th>
<th>Bus Group</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1, 2, 3</td>
</tr>
<tr>
<td><strong>Cubic</strong></td>
<td><strong>Model 1</strong></td>
</tr>
<tr>
<td>$c(x, \theta_1) = \theta_{11}x + \theta_{12}x^2 + \theta_{13}x^3$</td>
<td>$-131.063$</td>
</tr>
<tr>
<td></td>
<td>$-131.177$</td>
</tr>
<tr>
<td><strong>quadratic</strong></td>
<td><strong>Model 2</strong></td>
</tr>
<tr>
<td>$c(x, \theta_1) = \theta_{11}x + \theta_{12}x^2$</td>
<td>$-131.326$</td>
</tr>
<tr>
<td></td>
<td>$-131.534$</td>
</tr>
<tr>
<td><strong>linear</strong></td>
<td><strong>Model 3</strong></td>
</tr>
<tr>
<td>$c(x, \theta_1) = \theta_{11}x$</td>
<td>$-132.389$</td>
</tr>
<tr>
<td></td>
<td>$-134.747$</td>
</tr>
<tr>
<td><strong>square root</strong></td>
<td><strong>Model 4</strong></td>
</tr>
<tr>
<td>$c(x, \theta_1) = \theta_{11}\sqrt{x}$</td>
<td>$-132.104$</td>
</tr>
<tr>
<td></td>
<td>$-133.472$</td>
</tr>
<tr>
<td><strong>power</strong></td>
<td><strong>Model 5</strong></td>
</tr>
<tr>
<td>$c(x, \theta_1) = \theta_{11}x^{\theta_{12}}$</td>
<td>N.C.</td>
</tr>
<tr>
<td></td>
<td>N.C.</td>
</tr>
<tr>
<td><strong>hyperbolic</strong></td>
<td><strong>Model 6</strong></td>
</tr>
<tr>
<td>$c(x, \theta_1) = \theta_{11}/(91 - x)$</td>
<td>$-133.408$</td>
</tr>
<tr>
<td></td>
<td>$-138.894$</td>
</tr>
<tr>
<td><strong>mixed</strong></td>
<td><strong>Model 7</strong></td>
</tr>
<tr>
<td>$c(x, \theta_1) = \theta_{11}/(91 - x) + \theta_{12}\sqrt{x}$</td>
<td>$-131.418$</td>
</tr>
<tr>
<td></td>
<td>$-131.612$</td>
</tr>
<tr>
<td><strong>nonparametric</strong></td>
<td><strong>Model 8</strong></td>
</tr>
<tr>
<td>$c(x, \theta_1)$ any function</td>
<td>$-110.832$</td>
</tr>
<tr>
<td></td>
<td>$-110.832$</td>
</tr>
</tbody>
</table>

*a* First entry in each box is (partial) log likelihood value $\ell^2$ in equation (5.2) at $\beta = .9999$. Second entry is partial log likelihood value at $\beta = 0$.

*b* No convergence. Optimization algorithm attempted to drive $\theta_{12} \rightarrow 0$ and $\theta_{11} \rightarrow +\infty$. 
### TABLE IX
**STRUCTURAL ESTIMATES FOR COST FUNCTION** \( c(x, \theta_1) = .001\theta_1x \)

**Fixed Point Dimension** = 90
(Standard errors in parentheses)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimates/Log-Likelihood</th>
<th>Data Sample</th>
<th>Heterogeneity Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount Factor</td>
<td>Groups 1, 2, 3 3864 Observations</td>
<td>Group 4 4292 Observations</td>
<td>Groups 1, 2, 3, 4 8156 Observations</td>
</tr>
<tr>
<td>(\beta = .9999)</td>
<td>RC</td>
<td>11.7270 (2.602)</td>
<td>10.0750 (1.582)</td>
</tr>
<tr>
<td>(\theta_{11})</td>
<td>4.8259 (1.792)</td>
<td>2.2930 (0.639)</td>
<td>2.6275 (0.618)</td>
</tr>
<tr>
<td>(\theta_{30})</td>
<td>.3010 (.0074)</td>
<td>.3919 (.0075)</td>
<td>.3489 (.0052)</td>
</tr>
<tr>
<td>(\theta_{31})</td>
<td>.6884 (.0075)</td>
<td>.5953 (.0075)</td>
<td>.6394 (.0053)</td>
</tr>
<tr>
<td>(LL)</td>
<td>(-2708.366)</td>
<td>(-3304.155)</td>
<td>(-6055.250)</td>
</tr>
<tr>
<td>(\beta = 0)</td>
<td>RC</td>
<td>8.2985 (1.0417)</td>
<td>7.6358 (0.7197)</td>
</tr>
<tr>
<td>(\theta_{11})</td>
<td>109.9031 (26.163)</td>
<td>71.5133 (13.778)</td>
<td>70.2769 (10.750)</td>
</tr>
<tr>
<td>(\theta_{30})</td>
<td>.3010 (.0074)</td>
<td>.3919 (.0075)</td>
<td>.3488 (.0052)</td>
</tr>
<tr>
<td>(\theta_{31})</td>
<td>.6884 (.0075)</td>
<td>.5953 (.0075)</td>
<td>.6394 (.0053)</td>
</tr>
<tr>
<td>(LL)</td>
<td>(-2710.746)</td>
<td>(-3306.028)</td>
<td>(-6061.641)</td>
</tr>
<tr>
<td>Myopia test:</td>
<td>LR Statistic ((df = 1))</td>
<td>4.760</td>
<td>3.746</td>
</tr>
<tr>
<td>(\beta = 0) vs. (\beta = .9999)</td>
<td>Marginal Significance Level</td>
<td>0.0292</td>
<td>0.0529</td>
</tr>
</tbody>
</table>
This paper studies the value of patent protection: (i) what is the stochastic process determining the value of innovations?, (ii) how patent protection laws affect the decision to renew patents and the distribution of returns to innovation?

The model is an example of an **optimal stopping problem**. The model is setup with a finite horizon, but it does not have to be. Other examples: retirement, firm exit decisions, technology adoption, etc.

**Contributions:**

- Illustrate how we can infer the implicit option value of patents (or any other dynamic investment decision) from dynamic discrete choices (i.e. principle of revealed preference).
- This is done without actually observing profits or revenues from patents. Only the dynamic structure of renewal costs are needed.
- More technically, the paper is one of the firsts applications of simulation methods in econometrics (very influential).
Data and Institutional Details

- Three countries: France, Germany and UK
- Renewal date for all patents: $n_{m,t}(a) = \text{number of surviving patents at age } a \text{ in country } m \text{ from cohort } t$.
- Regulatory environment by country/cohort:
  - $f$: Number of automatic renewal years.
  - $L$: Expiration date on patent
  - $c = \{c_1, \ldots, c_T\}$: Deterministic renewal cost

### TABLE I
**Characteristics of the Data**

<table>
<thead>
<tr>
<th>Country Characteristic</th>
<th>France</th>
<th>U.K.</th>
<th>Germany</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $f$</td>
<td>2</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>2. $L$</td>
<td>20</td>
<td>16</td>
<td>18</td>
</tr>
<tr>
<td>4. First/last year in which renewals are observed</td>
<td>1970/81</td>
<td>1955/78</td>
<td>1955/74</td>
</tr>
<tr>
<td>5. Patents studied from cohort: all patents</td>
<td>Applied for</td>
<td>Applied for</td>
<td>Granted</td>
</tr>
<tr>
<td>6. Estimated average ratio of patents granted to patents applied for$^b$</td>
<td>.93</td>
<td>.83</td>
<td>.35</td>
</tr>
<tr>
<td>7. $NPAT = N/J$</td>
<td>36,865</td>
<td>37,286</td>
<td>21,273</td>
</tr>
</tbody>
</table>

$^a$ Additional notes on data collection and methodology.

$^b$ Adjustments for patent application rates and renewal rates.
Country differences in drop-out probabilities

Figure 2.—Average drop out proportions.
Country differences in renewal fee schedules

Figure 3.—Average of renewal fee schedules.
Consider the renewal problem for patent $i$

Stochastic sequence of returns from patent: $r_i = \{r_{i1}, \ldots, r_{iL}\}$

Evolution of returns depend on:

1. initial quality level
2. arrival of substitutes innovations that depreciate the value of the patent
3. arrival of complement innovations that increase its value.

Model structural parameters (per country):

- $\delta$ measures the normal obsolescence rate
- $\phi$ and $\sigma$ determines the arrival rate and magnitude of complementary innovations
- $\lambda$ determines to arrival rate of substitute innovations
- $\mu_0$ and $\sigma_0$ determines the initial quality pool of innovations

Discount factor $\beta$ is fixed.
Stochastic Process

- Markov process for returns:

\[ r_{it+1} = \tau_{it+1} \max\{\delta r_{it}, \xi_{it+1}\} \]

Where,

\[ \Pr(\tau_{it+1} = 0 | r_{it}, t) = \exp(-\lambda r_{it}) \]

\[ p(\xi_{it+1} | r_{it}, t) = \frac{1}{\phi^t \sigma} \exp \left( -\frac{\gamma + \xi_{it+1}}{\phi^t \sigma} \right) \]

\[ r_{i0} \sim LN(\mu_0, \sigma^2_0) \]

or more compactly for \( t > 0 \),

\[ f(r_{it+1} | r_{it}, t) = \begin{cases} 
\exp(-\lambda r_{it}) & \text{If } r_{it+1} = 0 \\
\Pr(\xi_{it+1} < \delta r_{it} | r_{it}, t) & \text{If } r_{it+1} = \delta r_{it} \\
\frac{1}{\phi^t \sigma} \exp \left( -\frac{\gamma + \xi_{it+1}}{\phi^t \sigma} \right) & \text{If } r_{it+1} > \delta r_{it} 
\end{cases} \]
Optimal stopping problem

- In the last year, the renewal value depends only on $c_L$ and $r_{iL}$:
  
  $$V(L, r_{iL}) = \max\{0, r_{iL} - c_L\}$$

  and therefore the patent is renewed if $r_{iL} > r_L^* = c_L$.

- At year $L - 1$, the value is defined recursively:

  $$V(L, r_{iL-1}) = \max\left\{0, r_{iL-1} - c_{L-1} + \beta \int_{r_L^*}^{\infty} V(L, r_{iL}) f(r_{iL}|r_{iL-1}, L-1) dr_{iL}\right\}$$

  This value function is strictly increasing in $r_{iL-1}$ (see proposition 1). Therefore, there exists a unique threshold such that the patent is renewed if

  $$r_{iL-1} > r_{L-1}^* = c_{L-1} - \beta \int_{r_L^*}^{\infty} V(L, r_{iL}) f(r_{iL}|r_{L-1}^*, L-1) dr_{iL}$$
Optimal stopping problem (continued)

- Similarly, for any year $t > 0$ the value function is defined recursively as follows:

$$V(L, r_{it}) = \max\{0, r_{it} - c_t + \beta \int_{r_{it+1}^*}^{\infty} V(t + 1, r_{it+1}) f(r_{it+1} | r_{it}, t) dr_{it+1}\}$$

which lead to a series of optimal stopping rules:

$$r_{it} > r_{it}^* = c_t - \beta \int_{r_{it+1}^*}^{\infty} V(t + 1, r_{it+1}) f(r_{it+1} | r_{it}^*, t) dr_{it+1}$$

- Given the function form assumptions on $f(r' | r_t, t)$, the thresholds can be solved analytically by backward induction.

- **Note:** When the terminal period is stochastic the value function becomes stationary (i.e. infinite horizon). For instance, optimal stopping problems arise when studying retirement or exit decisions:

$$V(s_t) = \max\left\{0, \pi(s_t) + \beta \int (1 - \delta(s_t)) V(s_{t+1}) f(s_{t+1} | s_t) ds_{t+1}\right\}$$
Estimation Method

- Likelihood of the observed renewal sequence $N_m$ conditional on the regulation environment $Z_m = \{L_m, f_m, c_m\}$ in country $m$:

$$L(N_m|Z_m, \theta) = \max_{\theta} \sum_{t=1}^{L} n_m(t) \ln \Pr(t^* = t|Z_m, \theta)$$

Where,

$$\Pr(t^* = t|\theta, Z_m) = \int_{-\infty}^{\infty} \int_{r_1^*}^{\infty} \int_{r_2^*}^{\infty} \cdots \int_{r_{t-1}^*}^{\infty} dF(r_{i1}, \ldots, r_{it-1}, r_{it}) dF_0(r_{i0})$$

- Monte Carlo integration approximation:

1. Sample $r_{i0}^s \sim \text{LN}(\mu_0, \sigma_0^2)$
2. Period 1:
   1. Sample $\tau_1^s$ from Bernoulli with probability $\exp(-\lambda r_0^s)$
   2. If $\tau_1^s = 1$, sample $\xi_1^s$ from exponential distribution. Otherwise, do not renew patent: $a_1^s = 0$.
   3. Calculate $r_1^1$
   4. Evaluate decision: $a_1^s = 1$ if $r_1^s > r_1^*$. 

... 

4. Repeat sampling for period $t$ if patent was renewed at $t - 1$
Estimation Method (continued)

- After collecting the simulated sequences of actions, we can evaluate the simulated choice-probability at period $t$:

$$\tilde{P}_S(t|\theta, Z_m) = \frac{1}{S} \sum_s 1(a^s_1 = 1, a^s_2 = 1, \ldots, a^s_{t-1} = 1, a^s_t = 0)$$

- Numerical problem: $\tilde{P}_S(t|\theta, Z_m)$ is not a smooth function of the parameters $\theta +$ equal to zero for some $t$ unless $S \to \infty$.

- Smooth alternative approximation:

$$\hat{P}_S(t, \theta, Z_m) = \frac{\exp\left(\tilde{P}_S(t|\theta, Z_m)/\eta\right)}{1 + \sum_{t'} \exp(\tilde{P}_S(t'|\theta, Z_m)/\eta)}$$

- Note: All the structural parameters are identified in this model (except $\beta$). The implicit normalization is that coefficient on renewal cost $c_t$ is one: all the parameters are expressed in dollar.
### TABLE II
**PARAMETER ESTIMATES**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Country</th>
<th>France</th>
<th>U.K.</th>
<th>Germany</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td></td>
<td>5689 (8.24)</td>
<td>5467 (6.09)</td>
<td>7460 (19.72)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td></td>
<td>9162 (13.67)</td>
<td>6919 (10.29)</td>
<td>8687 (17.09)</td>
</tr>
<tr>
<td>$\phi$</td>
<td></td>
<td>.5084 $(5.66 \times 10^{-4})$</td>
<td>.4383 $(2.17 \times 10^{-3})$</td>
<td>.4896 $(1.16 \times 10^{-3})$</td>
</tr>
<tr>
<td>$\delta$</td>
<td></td>
<td>.8475 $(2.62 \times 10^{-4})$</td>
<td>.8102 $(1.81 \times 10^{-3})$</td>
<td>.8861 $(2.48 \times 10^{-4})$</td>
</tr>
<tr>
<td>$\sigma_R$</td>
<td></td>
<td>1.579 $(2.92 \times 10^{-3})$</td>
<td>1.525 $(3.04 \times 10^{-3})$</td>
<td>1.158 $(2.36 \times 10^{-3})$</td>
</tr>
<tr>
<td>$\mu$</td>
<td></td>
<td>4.705 $(2.75 \times 10^{-3})$</td>
<td>5.425 $(2.55 \times 10^{-3})$</td>
<td>6.718 $(3.70 \times 10^{-3})$</td>
</tr>
<tr>
<td>$\theta$</td>
<td></td>
<td>.0990 $(6.36 \times 10^{-4})$</td>
<td>.36$^b$</td>
<td>.0855 $(2.46 \times 10^{-3})$</td>
</tr>
</tbody>
</table>

### B. Dimension$^c$

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Country</th>
<th>France</th>
<th>U.K.</th>
<th>Germany</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.1. $NPAT$</td>
<td></td>
<td>1,069,095</td>
<td>983,471</td>
<td>446,741</td>
</tr>
<tr>
<td>B.2. $NSIM$</td>
<td></td>
<td>20,000</td>
<td>20,000</td>
<td>20,000</td>
</tr>
<tr>
<td>B.3. Age: $f/L$</td>
<td></td>
<td>2/20</td>
<td>5/16</td>
<td>3/18</td>
</tr>
<tr>
<td>B.4. $NCHRT$</td>
<td></td>
<td>29</td>
<td>26</td>
<td>21</td>
</tr>
<tr>
<td>B.5. $NCHRTAGE$</td>
<td></td>
<td>238</td>
<td>272</td>
<td>237</td>
</tr>
</tbody>
</table>

### C. Summary Statistic$^d$

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Country</th>
<th>France</th>
<th>U.K.</th>
<th>Germany</th>
</tr>
</thead>
<tbody>
<tr>
<td>C.1. $MSE[\tilde{\pi}]$</td>
<td></td>
<td>$5.42 \times 10^{-4}$</td>
<td>$6.91 \times 10^{-4}$</td>
<td>$1.48 \times 10^{-4}$</td>
</tr>
<tr>
<td>C.2. $PDW[\tilde{\pi}]$</td>
<td></td>
<td>1.65</td>
<td>2.24</td>
<td>1.85</td>
</tr>
<tr>
<td>C.3. $V[\tilde{\pi}; data]$</td>
<td></td>
<td>$3.90 \times 10^{-2}$</td>
<td>$1.07 \times 10^{-2}$</td>
<td>$2.65 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

---

$^a$ Patents are assigned to cohorts by year of application. Numbers in parenthesis beside parameter estimates are their estimated standard errors.
### TABLE III

The Evolution of Implicit Revenues in the Early Ages

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>France</th>
<th>Germany</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(r_1</td>
<td>r_1 &gt; 0)$</td>
<td>380.43</td>
</tr>
<tr>
<td>Pr (Downside); Pr (Upside)</td>
<td>.0637; .1807</td>
<td>.0004; .2705</td>
</tr>
<tr>
<td>$\pi(2)$</td>
<td>.0637</td>
<td>(no required renewal)</td>
</tr>
<tr>
<td>$E(r_2</td>
<td>r_2 &gt; 0)$</td>
<td>1414.72</td>
</tr>
<tr>
<td>Pr (Downside); Pr (Upside)</td>
<td>.0387; .0331</td>
<td>.0006; .0584</td>
</tr>
<tr>
<td>$\pi(3)$</td>
<td>.0907</td>
<td>.0013</td>
</tr>
<tr>
<td>$E(r_3</td>
<td>r_3 &gt; 0)$</td>
<td>1432.24</td>
</tr>
<tr>
<td>Pr (Downside); Pr (Upside)</td>
<td>.0118; .0012</td>
<td>.0005; .0039</td>
</tr>
<tr>
<td>$\pi(4)$</td>
<td>.0792</td>
<td>.0121</td>
</tr>
<tr>
<td>$E(r_4</td>
<td>r_4 &gt; 0)$</td>
<td>1339.05</td>
</tr>
<tr>
<td>Pr (Downside); Pr (Upside)</td>
<td>.0048; 0.00</td>
<td>.0003; 0.00</td>
</tr>
<tr>
<td>$\pi(5)$</td>
<td>.0381</td>
<td>.0277</td>
</tr>
<tr>
<td>$E(r_5</td>
<td>r_5 &gt; 0)$</td>
<td>1192.70</td>
</tr>
<tr>
<td>NPAT</td>
<td>36,865</td>
<td>21,273</td>
</tr>
</tbody>
</table>
Summary of the Results

- Main differences across countries: (i) patent regulation rules, (ii) initial distribution of patent returns.
- Germany has a more selective screening system for granting new patents: higher mean and smaller variance of initial returns $r_{i0}$.
- Learning about complementary innovations: $\phi \approx 0.5$. Imply very fast learning/growth in returns.
- This has important policy implications: Regulator wants to keep initial renewing cost low, and increase them fast to extract rents from high value patents (low distortions after learning is over).
The distribution of realized patent value is highly skewed.

### TABLE V

**Percentiles (p1) and Lorenz Curve Coefficients (1c) from the Distribution of Realized Patent Values**

<table>
<thead>
<tr>
<th>Country</th>
<th>Per cent</th>
<th>$p_1$ ($)</th>
<th>$1c$ per cent</th>
<th>$p_1$ ($)</th>
<th>$1c$ per cent</th>
<th>$p_1$ ($)</th>
<th>$1c$ per cent</th>
</tr>
</thead>
<tbody>
<tr>
<td>France</td>
<td>.25</td>
<td>75.23</td>
<td>.544</td>
<td>355.55</td>
<td>.554</td>
<td>1,999.60</td>
<td>2.249</td>
</tr>
<tr>
<td></td>
<td>.50</td>
<td>533.96</td>
<td>1.833</td>
<td>1,516.84</td>
<td>3.247</td>
<td>6,252.93</td>
<td>7.341</td>
</tr>
<tr>
<td></td>
<td>.75</td>
<td>3,731.35</td>
<td>8.087</td>
<td>7,947.55</td>
<td>16.369</td>
<td>19,576.26</td>
<td>25.288</td>
</tr>
<tr>
<td></td>
<td>.85</td>
<td>10,292.06</td>
<td>19.575</td>
<td>15,357.09</td>
<td>31.721</td>
<td>32,428.14</td>
<td>41.001</td>
</tr>
<tr>
<td></td>
<td>.90</td>
<td>17,423.11</td>
<td>31.261</td>
<td>22,206.21</td>
<td>44.257</td>
<td>44,241.87</td>
<td>52.672</td>
</tr>
<tr>
<td></td>
<td>.95</td>
<td>31,609.59</td>
<td>52.461</td>
<td>34,740.07</td>
<td>62.960</td>
<td>65,753.61</td>
<td>69.223</td>
</tr>
<tr>
<td></td>
<td>.97</td>
<td>42,905.78</td>
<td>65.514</td>
<td>43,889.95</td>
<td>73.640</td>
<td>78,299.01</td>
<td>78.348</td>
</tr>
<tr>
<td></td>
<td>.98</td>
<td>51,215.84</td>
<td>73.729</td>
<td>51,277.22</td>
<td>80.072</td>
<td>94,842.63</td>
<td>83.800</td>
</tr>
<tr>
<td></td>
<td>.99</td>
<td>66,515.40</td>
<td>84.011</td>
<td>65,075.08</td>
<td>87.858</td>
<td>118,354.78</td>
<td>90.330</td>
</tr>
<tr>
<td>maximum</td>
<td></td>
<td>259,829.27</td>
<td>—</td>
<td>374,028.70</td>
<td>—</td>
<td>419,217.55</td>
<td>—</td>
</tr>
<tr>
<td>mean</td>
<td></td>
<td>5,631.03</td>
<td>—</td>
<td>7,357.05</td>
<td>—</td>
<td>16,169.48</td>
<td>—</td>
</tr>
<tr>
<td>$NPAT$</td>
<td></td>
<td>36,865</td>
<td>37,826</td>
<td></td>
<td>21,273</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

- Implied rate of returns on R&R: France = 15.56%, UK = 111.03%, Germany = 13.83%.
Sequential estimators of DDC models

**Key references:**

- Hotz and Miller (1993)
- Hotz, Miller, Sanders, and Smith (1994)
- Aguirregabiria and Mira (2002)

Consider the following dynamic discrete choice model with additively separable (AS) and conditional independent (CI) errors.

- A discrete actions.
- Payoff function: \( u(x|a) \)
- State space: \( (x, \epsilon) \).
- Where \( x \) is a discrete state vector, and \( \epsilon \) is an \( A \)-dimensions continuous vector.
- Distribution functions:
  - \( \Pr(x_{t+1} = x'|x_t, a) = f(x'|x, a) \)
  - \( g(\epsilon) \) is a type-1 EV density with unit variance.
Bellman Operator

Bellman equation:

\[ V(x) = \int \max_{a \in A} \left\{ u(x|a) + \epsilon(a) + \beta \sum_{x'} V(x') f(x'|x, a) \right\} g(\epsilon) d\epsilon \]

\[ = \int \max_{a \in A} \left\{ v(x|a) + \epsilon(a) \right\} g(\epsilon) d\epsilon \]

\[ = \ln \left( \sum_a \exp(v(x|a)) \right) + \gamma \]

\[ = \Gamma(V(x)) \]
Express $V(x)$ as a function of $P(a|x)$.

$$V(x) = \sum_a P(a|x) \ast \left\{ u(x|a) + E(\epsilon(a)|x, a) + \beta \sum_{x'} V(x') f(x'|x, a) \right\}$$

Where,

$$E(\epsilon(a)|x, a) = \frac{1}{P(a|x)} \int 1\left( v(x|a) + \epsilon(a) > v(x|a') + \epsilon(a'), a' \neq a \right) g(\epsilon) d\epsilon$$

$$e(a, P(a|x)) = \gamma - \ln P(a|x)$$
In Matrix form:

\[
V = \sum_a P(a) \ast \left[ u(a) + e(a, P) + \beta F(a) V \right]
\]

\[
\left[ I - \beta \sum_a P(a) \ast F(a) \right] V = \sum_a P(a) \ast \left[ u(a) + e(a, P) \right]
\]

\[
V(P) = \left[ I - \beta \sum_a P(a) \ast F(a) \right]^{-1} \left[ \sum_a P(a) \ast (u(a) + e(a, P)) \right]
\]

where \( F(a) \) is \(|X| \times |X|\) and \( V \) is \(|X| \times 1\).
CCP Operator (continued)

- In Matrix form:

\[ V = \sum_a P(a) \ast [u(a) + e(a, P) + \beta F(a) V] \]

\[ [I - \beta \sum_a P(a) \ast F(a)] V = \sum_a P(a) \ast [u(a) + e(a, P)] \]

\[ V(P) = [I - \beta \sum_a P(a) \ast F(a)]^{-1} \left[ \sum_a P(a) \ast (u(a) + e(a, P)) \right] \]

where \( F(a) \) is \(|X| \times |X|\) and \( V \) is \(|X| \times 1\).

- The CCP contraction mapping is:

\[
P(a|x) = \Pr \left( v(x|a, P) + \epsilon(a) > v(x|a', P) + \epsilon(a'), a' \neq a \right) \]

\[
= \frac{\exp (\tilde{v}(x|a, P))}{1 + \sum_{a' > 1} \exp (\tilde{v}(x|a, P))} \]

\[
= \Psi(a|x, P) \]

where \( \tilde{v}(x|a, P) = v(x|a, P) - v(x|1, P) \).
Two Special Cases

1. **Linear payoff:** If \( u(x|a, \theta) = x\theta \), the value function is also linear in \( \theta \).

\[
V(P) = Z(P)\theta + \lambda(P)
\]

Where

\[
Z(P) = \left( I - \beta \sum_a P(a) \ast F(a) \right)^{-1} \left[ \sum_a P(a) \ast X \right]
\]

\[
\lambda(P) = \left( I - \beta \sum_a P(a) \ast F(a) \right)^{-1} \left[ \sum_a P(a) \ast e(a, P) \right]
\]

---

**Absorbing state:** \( v(x|0) = 0 \) (e.g. Exit or retirement). This changes the value function:

\[
V(x, \varepsilon) = \max \begin{cases} 
 u(x) + \varepsilon(1) + \beta \sum_{x'} E_{\varepsilon'} \left[ V(x', \varepsilon') \right] & \text{if } x' \neq \bar{X} \\
 \bar{V}(x') & \text{if } x' = \bar{X} 
\end{cases}
\]

As before, the expected continuation value is:

\[
\bar{V}(x) = \log (1 + \exp(v(x))) + \gamma
\]
Two Special Cases

1. **Linear payoff:** If \( u(x|a, \theta) = x\theta \), the value function is also linear in \( \theta \).

\[
V(P) = Z(P)\theta + \lambda(P)
\]

Where

\[
Z(P) = \left[ I - \beta \sum_a P(a) \times F(a) \right]^{-1} \left[ \sum_a P(a) \times X \right]
\]

\[
\lambda(P) = \left[ I - \beta \sum_a P(a) \times F(a) \right]^{-1} \left[ \sum_a P(a) \times e(a, P) \right]
\]

2. **Absorbing state:** \( v(x|0) = 0 \) (e.g. Exit or retirement). This change the value function:

\[
V(x, \varepsilon) = \max \left\{ u(x) + \varepsilon(1) + \beta \sum_a E_{\varepsilon'} [V(x', \varepsilon')] \times F(x'|x), \varepsilon(0) \right\}
\]

As before, the expected continuation value is:

\[
\tilde{V}(x) = \log \left( \exp(0) + \exp \left( u(x) + \beta \sum_{x'} \tilde{V}(x') F(x'|x) \right) \right) + \gamma
\]

\[
= \log (1 + \exp(v(x))) + \gamma
\]
Two Special Cases (continued)

- The choice probability is given by:

$$\Pr(a = 1|x) = P(x) = \frac{\exp(v(x))}{1 + \exp(v(x))}$$

Note that the log of the “odds-ratio” is equal to the choice-specific value function:

$$\log \left( \frac{P(x)}{1 - P(x)} \right) = v(x)$$

- Therefore, the expected continuation value can be expressed as a function of $P(x)$:

$$\bar{V}^P(x) = \log (1 + \exp(v(x))) + \gamma = \log \left( 1 + \frac{P(s)}{1 - P(x)} \right) + \gamma$$

$$= - \log (1 - P(x)) + \gamma$$

- **Implication**: With an absorbing state, we don’t need to invert

$$[I - \beta \sum_a P(a) * F(a)]$$

to apply the CCP mapping.
Two-Step Estimator

- The objective is to estimate the structural parameters $\theta$ without repeatedly solving the DP problem
- **Initial step**: Reduced form of the model
  - Markov transition process: $\hat{f}(x'|x, a)$
  - Policy function: $\hat{P}(a|x)$
  - **Constraint**: Need to estimate both functions at EVERY state point $x$.

For finite samples, we need to impose smooth the policy function and interpolate between states are not visited (or infrequently). Kernels or local-polynomial techniques can be used.
Two-Step Estimator

- The objective is to estimate the structural parameters $\theta$ without repeatedly solving the DP problem

**Initial step:** Reduced form of the model

- Markov transition process: $\hat{f}(x'|x, a)$
- Policy function: $\hat{P}(a|x)$
- **Constraint:** Need to estimate both functions at EVERY state point $x$.

**How?** Ideally $\hat{P}(a|x)$ is estimated non-parametrically to avoid imposing a particular functional form on the policy function (i.e. no theory involved at this stage). This would correspond to a frequency estimator:

$$\hat{P}(a|x) = \frac{1}{n(x)} \sum_{i \in n(x)} 1(a_i = a)$$

- For finite samples, we need to impose smooth the policy function and interpolate between states are not visited (or infrequently). Kernels or local-polynomial techniques can be used.
Two-Step Estimator

- The objective is to estimate the structural parameters $\theta$ without repeatedly solving the DP problem
- **Initial step:** Reduced form of the model
  - Markov transition process: $\hat{f}(x'|x, a)$
  - Policy function: $\hat{P}(a|x)$
  - **Constraint:** Need to estimate both functions at EVERY state point $x$.
- How? Ideally $\hat{P}(a|x)$ is estimated non-parametrically to avoid imposing a particular functional form on the policy function (i.e. no theory involved at this stage). This would correspond to a frequency estimator:
  $$\hat{P}(a|x) = \frac{1}{n(x)} \sum_{i \in n(x)} 1(a_i = a)$$
- For finite samples, we need to impose smooth the policy function and interpolate between states are not visited (or infrequently). Kernels or local-polynomial techniques can be used.
- **Second-step:** Structural parameters conditional on $(\hat{P}, \hat{f})$
Example: Linear payoff function, $u(x|a, \theta) = x(a)\theta$

1- **Data Preparation:** Use $(\hat{P}, \hat{F})$ to calculate:

$$Z(\hat{P}, \hat{F}) = \left[I - \beta \sum_a \hat{P}(a) \ast \hat{F}(a)\right]^{-1} \left[\sum_a \hat{P}(a) \ast X(a)\right]$$

$$\lambda(\hat{P}, \hat{F}) = \left[I - \beta \sum_a \hat{P}(a) \ast F(a)\right]^{-1} \left[\sum_a \hat{P}(a) \ast e(a, \hat{P})\right]$$

2- **GMM:** Let $W_{it}$ denote a vector of predetermined instruments (e.g. state-variables and their interactions). We can construct moment conditions:

$$E \left(W_{it} \left[a_{it} - \Psi(a_{it}|x_{it}, \hat{P}, \hat{F})\right]\right) = 0$$

Where,

$$\Psi(a_{it}|x_{it}, \hat{P}, \hat{F}) = \frac{\exp\left(v(x_{it}|a_{it}, \hat{P}, \hat{F})\right)}{\sum_{a'} \exp(v(x_{it}|a', \hat{P}, \hat{F}))}$$

$$v(x|a, \hat{P}, \hat{F}) = x(a)\theta + \beta \sum_{x'} \underbrace{V(x|\hat{P}, \hat{F})}_{=Z(x, \hat{P}, \hat{F})\theta + \lambda(x, \hat{P}, \hat{F})} \hat{f}(x'|x, a).$$

$$v(x|a, \hat{P}, \hat{F}) = \left(x(a) + \beta \bar{Z}(x|\hat{P}, \hat{F})\right) \theta + \beta \bar{\lambda}(x|\hat{P}, \hat{F})$$

Therefore, the second-stage of problem is equivalent to a linear GMM (note: This also highlights the difficulty of identifying $\beta$ separately from $\theta$)
Pseudo-likelihood estimators (PML)

- **Source:** Aguirregabiria and Mira (2002)
- **Data:** Panel of $n$ individuals of $T$ periods:

$$(A, X) = \{a_{it}, x_{it}\}_{i=1,...,n; t=1,...,T}$$

- **2-Step estimator:**
  1. Obtain a flexible estimator of CCPs $\hat{P}^1(a|x)$
  2. Feasible PML estimator:

$$Q^{2S}(A, X) = \max_{\theta} \sum_{t} \sum_{i} \psi(a_{it}|x_{it}, \hat{P}^1, \hat{F}, \theta)$$

If $V(P)$ is linear, the second step is a linear probit/logit model.
Pseudo-likelihood estimators (PML)

- **NPL estimator:** The NPL repeat the PML and policy function iteration steps sequentially (i.e. swapping the fixed-point algorithm).
  1. Obtain a flexible estimator of CCPs \( \hat{P}^1(a|x) \)
  2. Feasible PML step:

\[
Q^{k+1}(A, X) = \max_{\theta} \sum_{t} \sum_{i} \Psi(a_{it}|x_{it}, \hat{P}^k, \hat{F}, \theta)
\]

3. Policy function iteration step:

\[
\hat{P}^{k+1}(a|x) = \Psi(a|x, \hat{P}^k, \hat{F}, \hat{\theta}^{k+1})
\]

4. Stop if \( ||\hat{P}^{k+1} - \hat{P}^k|| < \eta \), else repeat step 2 and 3.

- In the single agent case: The NPL is guaranteed to converge to the MLE estimator (i.e. NFXP).

- In practice, Aguirregabiria and Mira (2002) showed that 2 or 3 steps is sufficient to eliminate the small sample bias of the 2-step estimator, and is computationally easier to implement than the NFXP.
Simulation-based CCP estimator
Source: Hotz, Miller, Sanders, and Smith (1994)

- **Starting point:** The H&M GMM estimator suffers from a curse of dimensionality in $|X|$, since we must invert a $|X| \times |X|$ matrix to evaluate the continuation value (not true for optimal-stopping models). This is less severe for NFXP estimators, since we can use the value-function mapping to solve the policy functions.

- **Solution:**
  - First insight: We only need to know the relative choice-specific value function $\tilde{v}(a|x) = v(a|x) - v(1|x)$ to predict behavior.

  $$a_{it} = \begin{cases} 
  1 & \text{If } \tilde{v}(a|x) + \tilde{\epsilon}(a) < 0 \text{ for all } a \neq 1 \\
  a & \text{If } \max\{0, \tilde{v}(a'|x) + \tilde{\epsilon}(a')\} < \tilde{v}(a|x) + \tilde{\epsilon}(a) \text{ for all } a' \neq a 
  \end{cases}$$

  - Second insight: There exists a one-to-one mapping between $\tilde{v}(a|x)$ and $P(a|x)$. 
Simulation-based CCP estimator

- Logit example:

\[
P(a|x) = \frac{\exp(v(a|x))}{\sum_{a'} \exp(v(a'|x))} = \frac{\exp(\tilde{v}(a|x))}{1 + \sum_{a' > 1} \exp(\tilde{v}(a'|x))}
\]

⇔ \(\tilde{v}(a|x, P) = \ln P(a|x) - \ln P(1|x)\)

- Third insight: We can approximate the model’s predicted value function at any state \(x\) by simulating actions according to a policy function \(P(a|x)\).

\[
\hat{V}^S(x|P) = \frac{1}{S} \sum_s \sum_{\tau=0}^T \beta^\tau \left\{ u(x^s_{t+\tau}, a^s_{t+\tau}) + e(a^s_{t+\tau}|P(a^s_{t+\tau}|x^s_{t+\tau})) \right\}
\]

where \((x^s, a^s)\) is a simulated sequence of choices and states sampled from \(P(a|x)\) and \(f(x'|x, a)\), and \(e(a|P(a|x)) = E(\epsilon(a)|a_i = a, x, P)\) [closed-form expression]. Importantly, \(\lim_{S \to \infty} \hat{V}^S(x|P) = V(s|P)\).
Estimation Procedure

- **Step 1**: Estimate $\hat{P}(a|x)$ and $\hat{f}(x'|x,a)$, and compute the “dependent variable”:

$$\tilde{v}_n(a_{it}\mid x_{it}, \hat{P}) = \ln \hat{P}(a_{it}\mid x_{it}) - \ln \hat{P}(1\mid x_{it})$$

- **Step 2a**: Simulation of value functions of each observed state and choice $(x_{it}, a_{it})$. Each simulated sequence calculate the value of “future” choices:
  1. Calculate static value of $(x_{it}, a_{it})$: $u(x_{it}, a_{it}\mid \theta) + e(a_{it}\mid \hat{P}, x_{it})$
  2. Sample new state for period $t+1$: $x_{it+1} \sim \hat{f}(x'|x_{it}, a_{it})$
  3. Sample new choice for period $t+1$: $a_{it+1} \sim \hat{P}(a|x_{it})$

Repeat steps 1-3 for $T$ periods. This gives us the net present value of one simulated sequence:

$$v^s(a_{it}\mid x_{it}, \hat{P}, \theta) = u(x_{it}, a_{it}\mid \theta) + e(a_{it}\mid \hat{P}, x_{it+\tau})$$

$$+ \sum_{\tau=1}^{T} \beta^\tau \left[ u(x^s_{it+\tau}, a^s_{it+\tau}\mid \theta) + e(a^s_{it+\tau}\mid \hat{P}, x^s_{it+\tau}) \right]$$

Estimation of dynamic discrete choice models
Estimation Procedure (continued)

- Repeat this process $S$ times.
- This gives us the simulated value of choosing $a_{it}$ in state $x_{it}$:

$$v^S(a_{it}|x_{it}, \hat{P}, \theta) = \frac{1}{S} \sum_s v^s(a_{it}|x_{it}, \hat{P})$$

Let $\tilde{v}^S(a_{it}|x_{it}, \hat{P}, \theta) = v^S(a_{it}|x_{it}, \hat{P}, \theta) - v^S(1|x_{it}, \hat{P}, \theta)$.

- **Note:** If $u(x, a|\theta)$ is linear in $\theta$, we need to do this simulation process only once.

**Step 2b:** Moment conditions

$$E \left( W_{it} \left[ \tilde{v}_n(a_{it}|x_{it}, \hat{P}) - \tilde{v}^S(a_{it}|x_{it}, \hat{P}, \theta) \right] \right) = 0$$

where $W_{it}$ is a vector of instruments.
Estimation Procedure (continued)

- Importantly, setting up the moment conditions this way implies that the estimate will be consistent even with a finite number of simulated number of draws $S$.

- Why? The simulation error, $\tilde{v}(a_{it}|x_{it}, \hat{P}, \theta) - \tilde{v}^S(a_{it}|x_{it}, \hat{P}, \theta)$, is additive, and therefore vanishes as $n \to \infty$ (instead of $S \to \infty$).

- However, the small sample bias in $\hat{P}$ enters non-linearly in the moment conditions, and can induce severe biases (same as before):

  \[
  \ln \left( \hat{P}(a_{it}|x_{it}) + u_{it}(a) \right) - \ln \left( \hat{P}(1|x_{it}) + u_{it}(1) \right) \neq \ln \hat{P}(a_{it}|x_{it}) - \ln \hat{P}(1|x_{it}) + u_{it}
  \]

  For instance, if $\hat{P}(a_{it}|x_{it}) = 0$, the objective function is not defined.

- HMSS presents Monte-Carlo experiment to illustrate the small-sample bias. It can be quite large.


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